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L^p Spectral Theory of Kleinian Groups

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Let Γ be a geometrically finite group acting on the hyperbolic space $\mathbf{H}^{N+1}$ and let H_Γ denote the Laplace–Beltrami operator on $M=\Gamma\backslash\mathbf{H}^{N+1}$, which we assume is not compact. If M is either of finite volume or cusp-free, we determine completely the L^p spectrum of H_Γ for $1\leqslant p<\infty$, finding that it depends upon p in a nontrivial manner. We also obtain a number of pointwise and L^p decay properties of the L^2 eigenfunctions associated with eigenvalues E in the range $0\leqslant E<N^2/4$. © 1988 Academic Press, Inc.

1. INTRODUCTION

Let $\mathbf{H}^{N+1}$ denote the hyperbolic space

$$\{z=(x,y): x\in\mathbb{R}^N,\ y>0\} \tag{1.1}$$

with the metric

$$ds^2=y^{-2}(dx^2+dy^2) \tag{1.2}$$

and volume element

$$dv=y^{-N-1}\,dx\,dy. \tag{1.3}$$

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The Laplace–Beltrami operator is given by

$$Hf = -y^{N+1}\frac{\partial}{\partial y}\left(y^{1-N}\frac{\partial f}{\partial y}\right) - y^2 \Delta_x f, \tag{1.4}$$

where $\Delta_x = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_N^2$, initially defined on $C_0^\infty(\mathbf{H}^{N+1})$. As is well known [6, 16, 18], H is essentially selfadjoint and its closure H_2 on $L^2(\mathbf{H}^{n+1})$ has spectrum equal to $[N^2/4, \infty)$. The semigroup e^{-Ht} is a positivity-preserving contraction semigroup on $L^p(\mathbf{H}^{N+1})$ for $1 \leqslant p \leqslant \infty$ with $e^{-Ht}1 = 1$. If $1 \leqslant p < \infty$, then e^{-Ht} is strongly continuous on L^p and we denote its infinitesimal generator by H_p.

It has been shown by Hempel and Voigt [10, 11] that for a broad class of Schrödinger operators on Euclidean space, the spectrum of H_p is independent of p. However, as we show, in our case the spectrum of H_p does depend on p. In fact, we will show that it consists of the entire parabolic region P_p on and to the right of the curve

$$s \in \mathbb{R} \mapsto N^2 \frac{1}{p}\left(1 - \frac{1}{p}\right) + s^2 + isN\left(1 - \frac{2}{p}\right), \tag{1.5}$$

which degenerates to the interval $[N^2/4, \infty) \subset \mathbf{R}$ when $p = 2$.

We also investigate the spectrum of the Laplace operator on $M = \Gamma \backslash \mathbf{H}^{N+1}$ when Γ is a geometrically finite group of isometries of $\mathbf{H}^{N+1}$. We will assume, for simplicity, that Γ acts freely, and this assumption will be in effect throughout our paper. The analysis of the more general situation can readily be reduced to this case since it is known that any geometrically finite group of isometries of $\mathbf{H}^{N+1}$ has a subgroup of finite index which acts freely.

Under our assumption, M is a complete Riemannian manifold of constant negative curvature. We denote the induced Laplace–Beltrami operator by H_Γ. We always assume M is not compact; in the case of compact M, the questions we investigate obviously have very different answers. As above, we denote by $H_{\Gamma,2}$ the unique selfadjoint extension of H_Γ on $L^2(M)$, and by $H_{\Gamma,p}$ the generator of the associated heat semigroup on $L^p(M)$.

It is known [12] that the spectrum of $H_{\Gamma,2}$ is of the form

$$\{E_0, ..., E_m\} \cup [N^2/4, \infty), \tag{1.6}$$

where $\{E_0, ..., E_m\}$ is a finite set of eigenvalues of $H_{\Gamma,2}$, $E_j < N^2/4$. This set is possibly empty; whether it is depends on the exponent of convergence δ of Γ, which is characterized by

$$\delta(\Gamma) = \inf\left\{s \colon \sum_{\gamma \in \Gamma} e^{-s\rho(x, \gamma x)} < \infty\right\}, \tag{1.7}$$

$\rho(x, y)$ denoting the distance from x to y in the hyperbolic metric. Generally, $\delta(\Gamma) \in [0, N]$, and the discrete spectrum is nonempty if and only if $\delta(\Gamma) > N/2$. In this case, one has

$$\delta(N-\delta) = E_0 < E_1 \leqslant \cdots \leqslant E_m < N^2/4. \tag{1.8}$$

If M has finite volume, $H_{\Gamma, 2}$ may also have eigenvalues embedded in the interval $[N^2/4, \infty)$, but as shown in [12], if M has infinite volume this cannot occur.

In the case of $\delta \geqslant N/2$, we denote the corresponding eigenfunctions by φ_r, normalized by $\|\varphi_r\|_2 = 1$, and φ_0 is strictly positive on M. If M has finite volume, then $E_0 = 0$ and $\varphi_0 = (\text{vol}\, M)^{-1/2}$. More generally, the asymptotic behavior of φ_0 at infinity can be determined from the formula representing it as an integral over the limit set of Γ [15, 17, 18], but this approach does not work for the other eigenfunctions. We obtain a number of estimates on the eigenfunctions φ_r for $r \leqslant m$, particularly in Section 4.

Sections 2 and 3 are devoted to an analysis of the spectrum on $H_{\Gamma, p}$ on $L^p(M)$, for $1 \leqslant p < \infty$. We show that if M is either cusp-free or of finite volume, then

$$Sp(H_{\Gamma, p}) = \{E_0, ..., E_m\} \cup P_p. \tag{1.9}$$

An offshoot of that analysis is that the eigenfunction φ_r lies in $L^p(M)$ whenever E_r is outside the parabolic region P_p, i.e., provided

$$E_r < N^2 \frac{1}{p}\left(1 - \frac{1}{p}\right), \tag{1.10}$$

as long as M is either cusp-free or of finite volume. The estimates of Section 4 provide a different proof of this fact, which also works for more general geometrically finite quotients. We believe that the result (1.9) on the spectrum also holds for all geometrically finite groups.

Although we do not attempt to obtain any L^p analog of the detailed analysis of the continuous spectrum available in L^2 using the theory of Eisenstein series [13], such results would be of great interest.

2. L^p Theory for Cusp-Free Quotients

In this section we analyze the L^p spectrum of H_Γ acting on $L^p(M)$ when Γ is a geometrically finite group and M is of infinite volume with no cusps. We first state our result in the case when Γ is trivial.

THEOREM 1. *If* $1 \leqslant p < \infty$, *then* $Sp(H_p)$ *is precisely the parabolic region* P_p *defined in Section* 1.

This is a special case of Theorem 8, so we defer the proof. The first result we establish, on the way to proving Theorems 1 and 8, is the following.

LEMMA 2. *For all* $\varepsilon > 0$, *there exists a constant* $c_\varepsilon < \infty$ *such that* $\operatorname{Im} z \geqslant \frac{1}{2}N + \varepsilon$ *implies*

$$\left\| \left(H_1 - \frac{1}{4} N^2 - z^2 \right)^{-1} \right\|_{1,1} \leqslant c_\varepsilon. \tag{2.1}$$

In particular, $Sp(H_1)$ *is contained in the parabolic region* P_1.

Proof. Here, $\|T\|_{q,p}$ will denote the operator norm from L^p to L^q. We will give two proofs of Lemma 2, one using wave equation techniques, following similar results proved in [2, Sect. 3], and one of an "elementary" nature, using exact formulae for the resolvent kernel on $\mathbf{H}^{N+1}$. In both proofs, the goal will be to establish the estimate

$$\int_{\mathbf{H}^{N+1}} |G_z(x, y)| \, d\operatorname{vol}(x) \leqslant C_\varepsilon \tag{2.2}$$

for $z \in \mathbb{C}$ satisfying $\operatorname{Im} z \geqslant \frac{1}{2}N + \varepsilon$, where $G_z(x, y)$ is the Green function for the resolvent $(H - \frac{1}{4}N^2 - z^2)^{-1}$, defined on L^2 for $\operatorname{Im} z > 0$. We note that standard elliptic regularity results imply that $G_z(x, y)$ is in $L^1_{\text{loc}}(\mathbf{H}^{N+1})$ in x, for each fixed y, so it is only a matter of considering the behavior of the Green kernel as $x \to \infty$ in $\mathbf{H}^{N+1}$.

Our first method of estimation uses the representation

$$(K - z^2)^{-1} = (-2iz)^{-1} \int_{-\infty}^{\infty} e^{iz|t|} \cos t\sqrt{K} \, dt, \tag{2.3}$$

valid for $\operatorname{Im} z > 0$, where

$$K = H - \frac{1}{4} N^2. \tag{2.4}$$

By finite propagation speed for the solution operator $\cos t\sqrt{K}$ to the appropriate hyperbolic equation, we have

$$G_z(x, y) = (-2iz)^{-1} \int_{|t| \geqslant \rho} e^{iz|t|} \cos t\sqrt{K} \, \delta_y(x) \, dt, \tag{2.5}$$

where $\rho = \rho(x, y)$ in the hyperbolic distance between x and y, and δ_y is the Dirac delta function concentrated at y. While the details of the estimate of (2.5) are a special case of results of [2, Sect. 3], we will sketch them there, as they will illuminate further results. We will want to exploit the obvious operator norm estimate $\|\cos t\sqrt{K}\|_{2,2} \leqslant 1$. Since δ_y is not in L^2, it is convenient to replace it. The standard parametrix construction of elliptic PDE produces

$$G_z(x, y) = G_z^{(1)}(x, y) + G_z^{(2)}(x, y), \tag{2.6}$$

where $G_z^{(1)}$, the parametrix, is compactly supported (and L^1 in x for given y), and $G_z^{(2)}(x, y)$, the part it remains to estimate as $x \to \infty$, is of the form

$$G_z^{(2)}(x, y) = (K - z^2)^{-1} \omega(x), \tag{2.7}$$

where $\omega \in C_0^\infty(\mathbf{H}^{N+1})$ is supported in a ball centered at y, say of radius 1. Thus, for $\operatorname{Im} z > 0$,

$$G_z^{(2)}(x, y) = (-2iz)^{-1} \int_{|t| \geq \rho - 1} e^{iz|t|} (\cos t\sqrt{K})\, \omega(x)\, dt. \tag{2.8}$$

Thus, if $S_R = \{x \in \mathbf{H}^{N+1} : \rho(x, y) \geq R\}$, we have

$$\|G_z^{(2)}(\cdot, y)\|_{L^2(S_R)} \leq C|z|^{-1} e^{-(\operatorname{Im} z)R}. \tag{2.9}$$

If we look at the annular region

$$A_R = \{x \in \mathbf{H}^{N+1} : R \leq \rho(x, y) \leq R + 1\}, \qquad \operatorname{vol}(A_R) \sim e^{NR}, \tag{2.10}$$

then Cauchy's inequality yields

$$\begin{aligned} \|G_z^{(2)}(\cdot, y)\|_{L^1(A_R)} &\leq C|z|^{-1} e^{-(\operatorname{Im} z)R} (\operatorname{vol} A_R)^{1/2} \\ &\leq C|z|^{-1} e^{-(\operatorname{Im} z - (1/2)N)R}. \end{aligned} \tag{2.11}$$

Now summing over integral R yields the estimate (2.2) and completes the first proof of Lemma 2. ∎

The method outlined above has great flexibility; [2, Sect. 3] produced results for general Riemannian manifolds with bounded geometry, with some particular applications for rank one symmetric spaces, of which $\mathbf{H}^{N+1}$ is an example, dealing with some general classes of functions of the Laplace operator. The resolvent kernel for $\mathbf{H}^{N+1}$ can be treated on a more basic level, since one has simple exact formulas for the Green kernel in this situation. Indeed, for N even, there is the well known formula

$$G_z(x, y) = (-2iz)^{-1} [-(2\pi \sinh \rho)^{-1} \partial/\partial\rho]^k e^{iz\rho}, \qquad N = 2k. \tag{2.12}$$

When N is odd, one has the formula

$$\begin{aligned} G_z(x, y) = \sqrt{2}(-2iz)^{-1} \int_\rho^\infty & [-(2\pi \sinh s)^{-1} \partial/\partial s]^k e^{izs} \\ & \cdot (\cosh s - \cosh \rho)^{-1/2} \sinh s\, ds \end{aligned} \tag{2.13}$$

provided $N=2k-1$. In view of the result (2.10) on the volume of a shell in $\mathbf{H}^{N+1}$, the desired estimate (2.2) follows easily from these formulas, for $\operatorname{Im} z > \frac{1}{2}N$.

In view of the rotational symmetry of $\mathbf{H}^{N+1}$ about any point y, computation of $G_\lambda(x, y)$ can be reduced to a problem in ODE. See [20, pp. 269–270] for a discussion of $\mathbf{H}^2$, where $G_\lambda(x, y)$ is written in terms of Legendre functions; in this case (2.13) mirrors a well-known integral formula for Legendre functions.

It is also illuminating to tie together the two analyses of Lemma 2, as follows. Namely, there is a simple formula for the fundamental solution to the wave equation on $\mathbf{H}^{N+1}$, and via the following generalization of (2.3),

$$f(\sqrt{K}) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{f}(t) \cos t\sqrt{K}\, dt, \tag{2.14}$$

for an even function f, one can produce explicit formulas for numerous functions of the operator K, including the resolvent. Such formulas were emphasized in the papers of Lax and Phillips [12] and Cheeger and Taylor [3], which appeared almost simultaneously; see also monograph [19, Chaps. 4, 8]; these last two references also discuss analogous formulas for harmonic analysis on spheres. For the fundamental solution to the wave equation on $\mathbf{H}^{N+1}$, one has

$$K^{-1/2} \sin t\sqrt{K} = \lim_{\varepsilon \downarrow 0} -2A_N \operatorname{Im}[2\cos(it-\varepsilon) - 2\cosh\rho]^{-N/2}, \tag{2.15}$$

for a certain constant A_N, and the Schwartz kernel of $\cos t\sqrt{K}$ is obtained by taking the t-derivative. The formulas (2.12) and (2.13) can be obtained by using (2.14), which specializes to (2.3), and (2.15), with $f(\sqrt{K}) = (K - Z^2)^{-1}$. Note that, for N even, (2.15) is supported on the light cone $\rho = |t|$; this gives the strict Huygens principle in this case, and allows one to reduce (2.14) to the formula on $\mathbf{H}^{2k+1}$

$$f(\sqrt{K}) = (2\pi)^{-1/2}[-(2\pi \sinh\rho)^{-1}\, \partial/\partial\rho]^k \hat{f}(\rho), \tag{2.16}$$

where, as above, we identify an operator with its Schwartz kernel. On $\mathbf{H}^{2k}$, one has

$$f(\sqrt{K}) = \pi^{-1/2} \int_{\rho}^{\infty} [-(2\pi \sinh s)^{-1}\, \partial/\partial s]^k \hat{f}(s) \cdot (\cosh s - \cosh\rho)^{-1/2} \sinh s\, ds. \tag{2.17}$$

Of particular interest are the explicit formulas for the heat kernel. On $\mathbf{H}^{2k+1}$,

$$e^{-tK} = (4\pi t)^{-1/2}[-(2\pi \sinh\rho)^{-1}\partial/\partial\rho]^k\, e^{-\rho^2/4t}, \tag{2.18}$$

with an integral formula for $\mathbf{H}^{2k}$, specializing (2.17). The formulas (2.16)–(2.18) were also derived in [7], by a different method. These formulas for the heat kernel will play a role in some of our arguments below.

Having given an extended discussion of the Laplace operator on the full hyperbolic space, we begin the consider quotients. Our next result is valid for any group of isometries Γ acting on $\mathbf{H}^{N+1}$ with discrete orbits.

COROLLARY 3. *The bounds of Lemma 2 remain valid if we replace H_1 by the operator $H_{\Gamma,1}$ acting on $L^1(M)$.*

Proof. There is a natural quotient map from $L^1(\mathbf{H}^{N+1})$ onto $L^1(M)$ whose adjoint is the injection of $L^\infty(M)$ onto the space of Γ-invariant functions in $L^\infty(\mathbf{H}^{N+1})$. The identity

$$\mathrm{ex}(-H_{\Gamma,1}t)T = T\exp(-H_1 t) \tag{2.19}$$

is equivalent to the relation

$$K_\Gamma(t, z, \omega) = \sum_{\gamma \in \Gamma} K(t, z, \gamma\omega) \tag{2.20}$$

between the heat kernels on M and $\mathbf{H}^{N+1}$, respectively. Now (2.19) implies that

$$\|(H_{\Gamma,1} - \lambda)^{-1}\| \leqslant \|(H_1 - \lambda)^{-1}\| \tag{2.21}$$

for all $\lambda \notin Sp(H_1)$, so (2.1) implies the corresponding bound for $H_{\Gamma,1}$. ∎

Our next bound is essentially taken from [7]. We resume our hypothesis that M has no cusps.

PROPOSITION 4. *There exists a constant c_Γ such that*

$$\|e^{-H_\Gamma t}f\|_\infty \leqslant c(t)\|f\|_2 \tag{2.22}$$

for all $f \in L^2(M)$, where

$$c(t) = \begin{cases} c_\Gamma t^{-(N+1)/4} & \text{if } 0 < t \leqslant 1 \\ c_\Gamma e^{-\delta(N-\delta)t} & \text{if } 1 \leqslant t < \infty \end{cases} \tag{2.23}$$

provided the exponent of convergence δ lies in $[\frac{1}{2}N, N)$, the last bound being replaced by

$$c_\Gamma e^{-N^2t/4} \qquad \text{if} \quad \delta \in [0, \tfrac{1}{2}N].$$

Proof. If $0 < t \leqslant 1$, then the heat kernel for $\mathbf{H}^{N+1}$ satisfies the estimate

$$0 < K(t, z, w) \leqslant c_0 t^{-(N+1)/2} e^{-N\rho/2} e^{-\rho^2/4t}(1+\rho)^{N/2}, \tag{2.24}$$

as shown in [7], as a consequence of the formula (2.18) for N even, and the analogous integral formula for N odd, where ρ is the hyperbolic distance from z to ω. Thus,

$$0 < K_\Gamma(t, z, z) \leqslant c_1 t^{-(N+1)/2} \sum_{\gamma \in \Gamma} e^{-\rho(z, \gamma z)^2/4t}. \tag{2.25}$$

Since M has no cusps, the sum is bounded independently of z by [7, 15] and we deduce that

$$\begin{aligned} 0 < K_\Gamma(t, z, w) &\leqslant K_\Gamma(t, z, z)^{1/2} K_\Gamma(t, w, w)^{1/2} \\ &\leqslant c t^{-(N+1)/2}. \end{aligned} \tag{2.26}$$

Thus, $e^{-H_\Gamma t}$ is bounded from L^1 to L^∞ with

$$\|e^{-H_\Gamma t}\|_{\infty, 1} \leqslant c_2 t^{-(N+1)/2}. \tag{2.27}$$

Interpolation with the obvious bound from L^∞ to L^∞ yields

$$\|e^{-H_\Gamma t}\|_{\infty, 2} \leqslant c_3 t^{-(N+1)/4}. \tag{2.28}$$

When combined with the estimate $\|e^{-H_\Gamma t}\|_{2,2} \leqslant e^{-E_0 t}$, this yields the estimate of the proposition. ∎

We remark that estimates of the form (2.22) hold for the heat kernel whenever M is a complete Riemannian manifold with bounded geometry, as a consequence of the estimates of [2]. Among the quotients of hyperbolic space which are manifolds, precisely those which are cusp-free have bounded geometry, including injectivity radius bounded away from zero.

COROLLARY 5. *If M has no cusps and $\varphi \in L^2(M)$ satisfies $H_\Gamma \varphi = E\varphi$, then $\varphi \in L^\infty(M)$, and*

$$\|\varphi\|_\infty \leqslant C_\Gamma^1 \|\varphi\|_2.$$

Proof. We have

$$e^{-Et}\|\varphi\|_\infty = \|e^{-H_\Gamma t}\varphi\|_\infty,$$

so we can apply (2.22) *with* $t = 1/E$. ∎

We remark that if M posesses cusps but is of infinite volume, then Fourier analysis shows that φ_0 diverges to ∞ in each cusp. This is likely to happen for other eigenfunctions φ_r, though deciding whether such blow-up actually occurs can be a subtle matter.

Corollary 5 shows that, for M cusp-free, every L^2 eigenfunction belongs to L^p for all $p \in [2, \infty]$. The situation for $p \in [1, 2)$ is more interesting, and will be treated below. We are ready to establish our first result on the spectum of $H_{\Gamma, p}$.

LEMMA 6. *If* $1 \leqslant p \leqslant 2$, *then*

$$Sp(H_{\Gamma, p}) \subset \{E_0, ..., E_m\} \cup P_p. \tag{2.29}$$

Proof. If $1 \leqslant p \leqslant 2$, define L_1^p to the subspace of $L^p(M)$ defined by

$$\{f \in L^p(M): \langle f, \varphi_r \rangle = 0 \text{ for all } 0 \leqslant r \leqslant m\} \tag{2.30}$$

and define $L_2^p = L^p(M)/L_1^p$, so L_2^p has dimension $m+1$. Clearly L_1^p is invariant under $e^{-H_\Gamma t}$ and we define $H_{p,1}$ to be its generator. We let $H_{p,2}$ be the generator of the induced semigroup on the quotient space L_2^p. We claim that

$$Sp(H_{\Gamma, p}) = Sp(H_{p,1}) \cup Sp(H_{p,2}). \tag{2.31}$$

To show this, we note that if we define R_0: $L^p(M) \to L^p(M)$ by $R_0 = (H_{\Gamma, 1} + 1)^{-1}$, then R_0 leaves L_1^p invariant. If we define R_1 to be the restriction of R_0 to L_1^p and R_2 to be the induced operator on the quotient space L_2^p, then the spectral mapping theorem for generators of one-parameter contraction semigroups [5, p. 39] shows that (2.31) follows from

$$Sp(R_0) = Sp(R_1) \cup Sp(R_2). \tag{2.32}$$

In turn, (2.32) follows from the existence of an isomorphism

$$j: L^p(M) \approx L_1^p \oplus L_2^p$$

intertwining R with $R_1 \oplus R_2$, which can be arranged because L_2^p is finite dimensional.

Since we clearly have

$$Sp(H_{p,2}) = \{E_0, ..., E_m\} \tag{2.33}$$

for any $p \in [1, 2]$, it remains to determine the spectrum of $H_{p,1}$. We see by Corollary 3 that, for $\operatorname{Im} z \geqslant \frac{1}{2}N + \varepsilon$,

$$\left\| \left(H_{1,1} - \frac{1}{4} N^2 - z^2 \right)^{-1} \right\| \leqslant c_\varepsilon.$$

Since $Sp(H_{2,1}) \subset [\frac{1}{4}N^2, \infty)$, an application of the spectral theorem on the Hilbert space L_1^2 implies that

$$\left\| \left(H_{2,1} - \frac{1}{4} N^2 - z^2 \right)^{-1} \right\| \leqslant \varepsilon^{-2}$$

for $\operatorname{Im} z \geqslant \varepsilon > 0$. Then the Stein interpolation theorem implies

$$\left\| \left(H_{p,1} - \frac{1}{4} N^2 - z^2 \right)^{-1} \right\| < \infty$$

if $1 \leqslant p \leqslant 2$ and $\operatorname{Im} z > \lambda N/2$, where

$$\frac{1}{p} = \lambda + \frac{1-\lambda}{2}.$$

This gives the desired inclusion (2.29). ∎

The key to reversing the inclusion in (2.29) is provided by the following result.

LEMMA 7. *If $1 \leqslant p \leqslant q \leqslant 2$, then the boundary ∂P_q of P_q is contained in $Sp(H_{\Gamma,p})$.*

Proof. Since $e^{-H_\Gamma t}$ is a contraction on each $L^s(M)$, it follows from Proposition 4 that $e^{-H_\Gamma t}: L^r(M) \to L^s(M)$ whenever $1 \leqslant r \leqslant s \leqslant \infty$. Thus $Sp(H_{\Gamma,q}) \subset Sp(H_{\Gamma,p})$ by the argument of [10, Prop. 3.1], and we need only show that

$$\partial P_q \subset Sp(H_{\Gamma,q}). \tag{2.34}$$

We shall construct a sequence $f_n \in C_0^\infty(M)$ such that

$$\|H_{\Gamma,q} f_n - z f_n\|_q / \|f_n\|_q \to 0 \tag{2.35}$$

as $n \to \infty$, for each $z \in \partial P_q$, thus verifying that ∂P_q belongs to the approximate point spectrum of $H_{\Gamma,q}$. To achieve this, we take a region of the form $\Omega \times (0, \alpha)$ in the upper half $\mathbb{R}_+^{N+1}$, which lies entirely within one fundamental domain of Γ, using the fact $\Gamma \backslash \mathbf{H}^{N+1}$ has at least one end. If $z \in \partial P_q$, write $z = ((N/q) + is)(N - (N/q) - is)$ for some $s \in \mathbf{R}$. If we put

$$f_n(x, y) = b(x)\, c_n(\log y)\, y^{(N/q) + is} \tag{2.36}$$

for $x \in \mathbf{R}^n$, $y \in \mathbf{R}^+$, where $b \in C_0^\infty(\Omega)$ and $c_n \in C_0^\infty(-\infty, \log \alpha)$, then a direct computation shows that, for any choice of b, we make (2.35) as small as we like by an appropriate choice of c_n. ∎

We now give the major result of this section.

THEOREM 8. *If $1 \leqslant p < \infty$ and M has no cusps, then*

$$Sp(H_{\Gamma,p}) = \{E_0, ..., E_m\} \cup P_p. \tag{2.37}$$

If an eigenvalue E_r lies outside the parabolic region P_p, then the corresponding eigenfunction φ_r lies in $L^p(M)$.

Proof. If $1 \leqslant p \leqslant 2$, then Lemma 6 shows that $Sp(H_{\Gamma,p}) \subset \{E_0, ..., E_m\} \cup P_p$, and its proof, involving (2.31) and (2.33), shows $\{E_0, ..., E_m\} \subset Sp(H_{\Gamma,p})$. Since Lemma 7 gives $P_p \subset Sp(H_{\Gamma,p})$, we have the identity (2.37) for $p \in [1, 2]$. If $p \in [2, \infty)$, then both sides of (2.37) are invariant under the map $p \mapsto q$ where $(1/p) + (1/q) = 1$, so the identity (2.37) still holds.

We have already shown that each eigenfunction φ_r belongs to $L^p(M)$ for $p \in [2, \infty]$, so the final statement of the theorem needs to be established only for $1 \leqslant p \leqslant 2$. If E_r is outside the parabolic region, then it is an isolated point of the spectrum of $H_{\Gamma,p}$ and so the spectral calculus allows us to construct a corresponding spectral projection P_r which is a bounded operator on $L^p(M)$:

$$P_{r,p} = (2\pi i)^{-1} \int_\gamma (\lambda - H_{\Gamma,p})^{-1}\, d\lambda, \tag{2.38}$$

where γ is a small circle encircling E_r. From this formula it is clear that, as long as P_p does not include E_r,

$$P_{r,p} = P_{r,2} \text{ on } L^p(M) \cap L^2(M). \tag{2.39}$$

Suppose that the range of $P_{r,2}$ has dimension l; i.e., l of the E_j are equal to E_r. One pick an orthonormal basis of this range and approximate each element of such a basis closely in the L^2 norm by elements $\psi_j \in C_0^\infty(M)$, $1 \leqslant j \leqslant l$. If $V \subset C_0^\infty(M)$ denotes the linear span of these ψ_j, then $P_{r,2}$ maps V isomorphically onro its range, and hence the identity (2.39) implies this range lies in $L^p(M)$, provided $E_r \notin P_p$. This completes the proof of Theorem 8. ∎

We remark that another way of stating this last result is that if $\varphi \in L^2(M)$ and $H_{\Gamma,2}\varphi = E\varphi$, for some $E < N^2/4$, then

$$\varphi \in L^p(M) \qquad \text{for all } p \in (2(1 + (1 - 4E/N^2)^{1/2})^{-1}, \infty]. \tag{2.40}$$

In the case of the smallest eigenvalue $E_0 = \delta(N - \delta)$, for M with exponent of convergence $\delta \in (N/2, N)$, this implies

$$\varphi_0 \in L^p(M) \qquad \text{for all } p \in (N/\delta, \infty]. \tag{2.41}$$

Estimates given in Section 4 will yield a second proof of (2.40).

3. L^p THEORY FOR FINITE VOLUME QUOTIENTS

In this section we consider the analogs of the theorems of Section 2 where Γ is geometrically finite and M is of finite volume. In this case, $E_0 = 0$ and the corresponding eigenfunction φ_0 is constant, but there is no reason to believe that the other eigenfunctions $\varphi_r \in L^2$ are bounded on M.

THEOREM 9. *If $1 \leqslant p < \infty$ and M has finite volume, then*

$$Sp(H_{\Gamma,p}) = \{E_0, ..., E_m\} \cup P_p. \tag{3.1}$$

If $0 \leqslant r \leqslant m$, then

$$\varphi_r \in L^p(M) \qquad \textit{for all} \quad p \in [1, 2\{1-(1-4E_r/N^2)^{1/2}\}^{-1}). \tag{3.2}$$

Proof. We indicate how the proof differs from that of Section 2. Since $\operatorname{vol} M < \infty$, $L^q(M)$ is continuously embedded in $L^p(M)$ whenever $1 \leqslant q \leqslant p$. This implies that $\varphi_r \in L^p$ for all $p \in [1, 2]$. One sees, as in Lemma 6, that

$$Sp(H_{\Gamma,p}) \subset \{E_0, ..., E_m\} \cup P_p,$$

provided $2 \leqslant p < \infty$. In order to conclude, as in Lemma 7, that $\partial P_q \subset Sp(H_{\Gamma,p})$ whenever $2 \leqslant q \leqslant p < \infty$, we make the following comments.

By the use of an appropriate isometry of $\mathbf{H}^{N+1}$, one may assume that one of the cusps of a fundamental domain D of Γ ends at ∞. By the geometric finiteness of Γ, one can arrange that the intersection of D with $\mathbf{R}^N \times (\alpha, \infty)$ coincide with $\Omega \times (\alpha, \infty)$, for some $\alpha > 0$ and some polyhedral region $\Omega \subset \mathbf{R}^N$. Then choosing $f_n \in C_0^\infty(M)$ of the form (2.36) with $b \equiv 1$ and c_n an appropriate sequence in $C_0^\infty(\log \alpha, \infty)$ shows that $((N/q) + is)(N - (N/q) - is)$ lies in the approximate point spectrum of $H_{\Gamma,q}$. The remainder of the proof follows Section 2 exactly. ∎

We remark that it is natural to conjecture that if Γ is a general geometrically finite group, then (3.1) still holds. In any event, it is easy to modify the proof of Lemma 7 to show that ∂P_p belongs to the approximate spectrum of $H_{\Gamma,p}$ in general.

We also remark that a second proof of (3.2) will also from estimates established in Section 4. Furthermore, we will show in Section 4 that, for general geometrically finite Γ, any eigenfunction φ_r with eigenvalue $E_r < N^2/4$ belongs to L^p as long as p satisfies both of the conditions in (2.40) and (3.2).

4. Further Bounds on the L^2-Eigenfunctions

In this section we obtain second proofs of results on L^2-eigenfunctions of H_Γ contained in Theorems 8 and 9, some generalizations of these results, and some further estimates exploiting weighted L^2-estimates techniques, some of which go back to a theme in the theory of Schrödinger operators [14, 8, 1]. Our first results use wave equation techniques such as exploited in [2]. These results work for any geometrically finite Γ. We recall that, in this context,

$$Sp(H_{\Gamma,2}) = \{E_0, ..., E_m\} \cup [N^2/4, \infty), \tag{4.1}$$

where $E_0 < E_1 \leqslant \cdots E_m$ is a finite set of eigenvalues, repeated according to multiplicity, all $< N^2/4$.

Lemma 10. *There exists a compact set $K \subset M$ such that*

$$\langle H_\Gamma f, f\rangle \geqslant (N^2/4)\|f\|_2^2 \tag{4.2}$$

for all $f \in C_0^\infty(M-K)$.

Proof. Such a result is true for any complete Riemannian manifold M whose Laplace operator has spectrum satisfying (4.1). This can be proven as follows. If it were not true, there would exist an infinite sequence $f_n \in C_0^\infty(M)$ with disjoint supports such that $\langle H_\Gamma f_n, f_n\rangle < (N^2/4)\|f_n\|_2^2$. By restricting the quadratic form of H_Γ to the linear span of $\{f_n\}$ we obtain a contradiction to the fact that H_Γ has only a finite number of eigenvalues below $N^2/4$. ∎

Using Lemma 10, we obtain our first key result on the behavior of an eigenfunction φ_r corresponding to eigenvalue $E_r < N^2/4$, for any geometrically finite M.

Proposition 11. *If $a \in M$ and*

$$S(R) = \{x \in M: \rho(x, a) > R\}, \tag{4.3}$$

then there exists a constant C such that

$$\int_{S(R)} |\varphi_r|^2 \, dv \leqslant C \exp[-R\sqrt{N^2 - 4E_r}], \tag{4.4}$$

for all $R > 0$.

Proof. Pick a compact set K satisfying (4.2), pick $\varphi \in C_0^\infty(M)$ equal to 1 on a neighborhood of K, and set $\psi = 1 - \varphi$. Set $\psi_r = 4\varphi_r$ and let H'_Γ be the negative Laplacian on $M-K$ with Dirichlet boundary conditions. Then ψ_r belongs to the domain of H'_Γ and

$$(H'_\Gamma - E_r)\,\psi_r = f \in C_0^\infty(M-K). \tag{4.5}$$

Furthermore,

$$Sp(H'_\Gamma) \subset [N^2/4, \infty). \tag{4.6}$$

It suffices to prove the bound (4.4) with φ_r replaced by ψ_r. We do this by applying wave equation techniques within $L^2(M-K)$ as in [2]. If

$$A = (H'_\Gamma - N^2/4)^{1/2}$$

and

$$l = (N^2/4 - E_r)^{1/2},$$

then

$$\begin{aligned}\psi_r &= (A^2 + l^2)^{-1} f \\ &= (2l)^{-1} \int_{-\infty}^{\infty} e^{-l|t|} (\cos tA) f\, dt.\end{aligned} \tag{4.7}$$

The finite propagation speed of the wave equation implies that for a suitable constant a and all $R > a$,

$$\psi_r|_{S(R)} = (2l)^{-1} \int_{|t| > R-a} e^{-l|t|} (\cos tA) f|_{S(R)}\, dt. \tag{4.8}$$

Hence

$$\|\psi_r\|_{L^2(S_r)} \leqslant l^{-2} \|f\|_2\, e^{-l(R-a)}, \tag{4.9}$$

which proves the proposition. ∎

We remark that the estimate (4.4) implies, but is slightly sharper than, the weighted L^2 estimate

$$\int |\varphi_r|^2\, e^{(1-\varepsilon)\sqrt{N^2-4E_r}\,\rho}\, dv < \infty, \tag{4.10}$$

an estimate which also follows from the work of Agmon [1]. We now show how such an estimate yields a proof of a generalization of the result (2.40).

PROPOSITION 12. *For any geometrically finite Γ, if φ_r is an eigenfunction of $H_{\Gamma,2}$ with eigenvalue $E_r < N^2/4$, then*

$$\varphi_r \in L^p(M) \qquad \textit{for} \quad p \in (2(1+(1-4E_r/N^2)^{1/2})^{-1}, 2]. \tag{4.11}$$

Proof. By Cauchy's inequality, for $p \leqslant 2$,

$$\int |\varphi_r|^p \, dv \leqslant \| |\varphi|^p \, e^{(1-\varepsilon)(p/2)\sqrt{N^2-4E_r}\,\rho} \|_{L^{2/p}} \, \| e^{-(1-\varepsilon)(p/2)\sqrt{N^2-4E_r}} \|_{L^q} \tag{4.12}$$

with $(p/2)+(1/q)=1$, i.e., $q=2/(2-p)$. The first factor on the right is simply the $p/2$ power of (4.10), which is finite. It remains to examine the finiteness of the second factor on the right side of (4.12). Indeed,

$$\begin{aligned} &\int_M e^{-(1-\varepsilon)(pq/2)\sqrt{N^2-4E_r}\,\rho} \, dv \\ &\quad \leqslant \int_{\mathbf{H}^{N+1}} e^{-(1-\varepsilon)(pq/2)\sqrt{N^2-4E_r}\,\rho} \, dv \\ &\quad \leqslant c \int_0^\infty \exp[N-(1-\varepsilon)(pq/2)\sqrt{N^2-4E_r}] \, \rho \, d\rho. \end{aligned}$$

This is finite as long as the exponent is negative, i.e., provided

$$pq > 2/\sqrt{1-4E_r/N^2}.$$

In view of the relation between p and q, this establishes (4.11). ∎

We next show how Proposition 11 leads to an alternate proof of the result (3.2) on an eigenfunction φ_r when M has finite volume, and indeed, to a pointwise estimate on such φ_r which could generally be expected to be sharp.

PROPOSITION 13. *Let Γ be geometrically finite, and suppose $M = \Gamma \backslash \mathbf{H}^{N+1}$ has finite volume. Let $H_\Gamma \varphi_r = E_r \varphi_r$ with $\varphi_r \in L^2(M)$, $E_r < N^2/4$. Fix $a \in M$. Then*

$$|\varphi_r(x)|^2 \leqslant C \exp[(N-\sqrt{N^2-4E_r})\, \rho(x,a)]. \tag{4.13}$$

Proof. Since elliptic regularity theorems imply that φ_r is smooth, it is sufficient to estimate φ_r in a cusp, which we can take to be of the form $\Omega \times (\gamma, \infty)$, with $\Omega \subset \mathbf{R}^N$ a polyhedron. If we take $a=(0,1) \in \mathbf{R}^N \times \mathbf{R}^+$, then (4.13) is equivalent to the estimate

$$|\varphi_r(x,y)|^2 \leqslant C y^\alpha, \qquad \alpha = N - \sqrt{N^2-4E_r} \tag{4.14}$$

within the cusp. If we regard φ_Γ as a Γ-invariant function on $\mathbf{H}^{N+1}$ and let B be the hyperbolic ball with center $(0, y)$ and radius 1, then the number of fundamental domains meeting B is of order y^N as $y \to \infty$, so Proposition 12 implies

$$\int_B |\varphi_r|^2 \, dv \leqslant c_0 \, y^N \cdot y^{-\sqrt{N^2-4E_r}}. \tag{4.15}$$

Since $H\varphi_r = E_r \varphi_r$, a standard Sobolev estimate yields (4.14), and proves the proposition. ∎

Note that if D denotes such a cusp in a fundamental domain as described above, then

$$\int_D |\varphi_r(x)|^p \, dv \leqslant c \int_\gamma^\infty y^{(p/2)(N-\sqrt{N^2-4E_r})-N-1} \, dy, \tag{4.16}$$

which is finite as long as $(p/2)(N-\sqrt{N^2-4E_r})-N<0$, i.e., provided $p<2/[1-\sqrt{1-4E_r^2/N^2}]$, so we obtain our second proof of the result (3.2).

We remark that Proposition 13 can be established by classical separation of variables methods. Indeed, in the cusp D, the eigenfunction $\varphi_r(x, y)$ has the expansion

$$\varphi_r(x, y) = c_0 \, y^{(N-\sqrt{N^2-4E_r})/2} + \sum_{j \geqslant 1} c_j u_j(x) \, v_j(y), \tag{4.17}$$

where u_j are the nonconstant eigenfunctions of Δ_x on the polyhedron Ω, with periodic boundary conditions, say $\Delta_x u_j = -\mu_j^2 u_j$, and $v_j(y)$ is a solution to the ODE

$$v_j''(y) + \frac{1-N}{y} v_j'(y) + \left(\frac{E_r}{y^2} - \mu_j^2\right) v_j(y) = 0. \tag{4.18}$$

In the special case of a cusp for a quotient of $\mathbf{H}^2$, one has the Fourier expansion of Maass waveforms, described in [20, p. 208]. Analysis of (4.17) and (4.18) also yields the estimate (4.14) of Proposition 13. Only exceptionally would the coefficient c_0 be expected to vanish, so such an estimate is seen to be sharp in a certain sense. The proof of Proposition 13 by such separation of variable techniques is not shorter than that presented above, but it is of a classical flavor, and is likely to be known. A further advantage of such expansions is to produce results on eigenfunctions associated with $H_{\Gamma,2}$ embedded in the continuum $[N^2/4, \infty)$. See, for example, some results along these lines in [20, pp. 225–227].

We next produce pointwise estimates valid for general geometrically finite domains.

PROPOSITION 14. *Let φ_r be a normalized L^2-eigenfunction for H_Γ on $\Gamma\backslash\mathbf{H}^{N+1}$ with eigenvalue $E_r < N^2/4$. For $x \in \mathbf{H}^{N+1}$, $R > 0$, let $B_R(x)$ be the ball in $\mathbf{H}^{N+1}$ centered at x, and of radius R; suppose $B_R(x)$ intersects at most $\nu(x, R)$ fundamental domains of Γ. Suppose you know that*

$$\varphi_r \in L^p(\Gamma\backslash\mathbf{H}^{N+1}), \tag{4.19}$$

for some $p \in [1, 2/[1 - \sqrt{1 - 4E_r/N^2}])$. Then we have the estimate

$$|\varphi_r(x)| \leqslant C\nu(x, R)^{1/p}\, e^{-[(1/2)\sqrt{N^2 - 4E_r} + (N/p) - (N/2)]R}. \tag{4.20}$$

For the proof, let $g(\rho)$ be the function obtained by averaging $\varphi_r(x)\,\varphi_r$ under the group of rotations about x. This function has the following obvious properties:

$$g(\rho) \text{ is a smooth even function of } \rho, \text{ defined for all } \rho \in \mathbf{R}, \tag{4.21}$$

$$g(0) = 1, \tag{4.22}$$

$$g''(\rho) + N(\coth \rho)\, g'(\rho) = -E_r\, g(\rho), \qquad \text{for} \quad \rho \in \mathbf{R}\backslash 0. \tag{4.23}$$

It will be useful to know the following facts, which also play a role in arguments in [18].

LEMMA 15. *The properties* (4.21)–(4.23) *uniquely characterize $g(\rho)$. One has $g(\rho) > 0$ for all $\rho \in \mathbf{R}$; more precisely, for constants $c_j > 0$,*

$$c_1 e^{-\alpha\rho} \leqslant g(\rho) \leqslant c_2 e^{-\alpha\rho}, \tag{4.24}$$

with

$$\alpha = \frac{N}{2} - \frac{1}{2}\sqrt{N^2 - 4E_r}. \tag{4.25}$$

Proof. Suppose $g(\rho_0) = 0$. Then $g(\rho)$ defines an eigenfunction for $-\Delta$ on the hyperbolic ball $B_{\rho_0}(x)$ with Dirichlet boundary conditions, with eigenvalue E_r, contradicting the obvious fact that the spectrum of such an operator is contained in $[N^2/4, \infty)$. Since (4.21)–(4.23) imply $g(\rho)$ is real-valued, this gives $g(\rho) > 0$ for all ρ, and this fact easily yields uniqueness. As for the behavior (4.24), since $\coth \rho \sim 1$ for $\rho \to +\infty$, the standard asymptotic theory of ODE [4, 9] implies $g(\rho) = Ag_1(\rho) + Bg_2(\rho)$, where g_1 and g_2 are two linearly independent solutions to the ODE (4.23), satisfying

$$g_1(\rho) \sim e^{-\alpha\rho}, \quad g_2(\rho) \sim e^{-\beta\rho}, \qquad \rho \to +\infty, \tag{4.26}$$

with α given by (4.25) and

$$\beta = \frac{N}{2} + \frac{1}{2}\sqrt{N^2 - 4E_r}. \tag{4.27}$$

The coefficient A of $g_1(\rho)$ must be nonvanishing, since otherwise g would belong to $L^2(\mathbf{H}^{N+1})$, producing an impossible eigenfunction for the Laplace operator on $\mathbf{H}^{N+1}$. Since we already know $g(\rho) > 0$, this implies (4.24). ∎

Using this lemma, we can prove Proposition 14. We know from Proposition 12 that $\varphi_r \in L^p(\Gamma\backslash\mathbf{H}^{N+1})$ for all $p \in (N/\beta, 2]$, with β given by (4.27). For p in this range, we have

$$\begin{aligned} |\varphi_r(x)|^p \int_0^R |g(\rho)|^p \sinh^N \rho \, d\rho &\leqslant \|\varphi_r\|^p_{L^p(B_R(x))} \\ &< v(x, R)\, \|\varphi_r\|^p_{L^p(M)}. \end{aligned} \tag{4.28}$$

The lower estimate in (4.24) implies that the integral on the left side of (4.28) is $\geqslant ce^{(N-\alpha p)R}$, provided $R \geqslant 1$ and $p < n/\alpha$. This gives the estimate (4.20).

If we rewrite (4.20) as

$$|\varphi_r(x)| \leqslant c[v(x, R)\, e^{-NR}]^{1/p}\, e^{(1/2)[N - \sqrt{N^2 - 4E_r}]R}, \tag{4.20$'$}$$

it is clear that we will want to take p as large as possible if $v(x, R)\, e^{-NR} \geqslant 1$ and as small as possible if $v(x, R)\, e^{-NR} \leqslant 1$. In particular, suppose $M = \Gamma\backslash H^{N+1}$ is cusp-free. In that case, fix $x_0 \in \mathbf{H}^{N+1}$, identified with a point in M, and fix $a > 0$. Then for any x in a fundamental domain of Γ in $\mathbf{H}^{N+1}$, if $\rho(x_0, x) = R + a$, there is a bound on $v(x, R)$, and we have the following.

COROLLARY 16. *If $M = \Gamma\backslash\mathbf{H}^{N+1}$ is cusp-free, then for each $\varepsilon > 0$*

$$|\varphi_r(x)| \leqslant C_\varepsilon e^{-(1-\varepsilon)\sqrt{N^2 - 4E_r}\,\rho(x, x_0)}. \tag{4.29}$$

Proof. Apply (4.20) with p arbitrarily close to $2/[1 + \sqrt{1 - 4E_r/N^2}]$, which we know can be done by Proposition 12, or alternatively by (2.40). ∎

We note that (4.29) is substantially better than one would get by applying a Sobolev inequality to (4.4) in a straightforward manner, which would produce only an estimate of the form

$$|\varphi_r(x)| \leqslant ce^{-(1/2)\sqrt{N^2 - 4E_r}\,\rho(x, x_0)}.$$

On the other hand, it is not clear that (4.29) is sharp. If we knew that $\varphi_r(x)$ were evenly spread out over each shell $\rho(x, x_0) = R$, one would expect an estimate of the form

$$|\varphi_r(x)| \leqslant ce^{-(1/2)(N+\sqrt{N^2-4E_r})\,\rho(x,x_0)}. \tag{4.30}$$

We do not know how to preclude the concentration of $\varphi_r(x)$ on a relatively small set, so we cannot improve the estimate (4.29).

If one were to apply (4.20) to the case when $M = \Gamma\backslash\mathbf{H}^{N+1}$ has finite volume, with $\rho(x, x_0) = R + a$ (a constant), then, out along a cusp one has $v(x, R)$ fairly large, and the estimate so obtained is obviously poorer than our estimate (4.13). Part of the reason for the deficiency is that in (4.28) we neglected to take account of the fact that $\|\varphi_r\|^p_{L^p(B_R(x))}$ could be a good deal smaller than $v(x, R)\,\|\varphi_r\|^p_{L^p(M)}$. In fact, with R fixed, say $R = 1$ and $p = 2$, we can use the estimate (4.4); this basically duplicates the argument proving the estimate (4.13) in Proposition 13.

We can improve Proposition 14 to include a treatment of this last case as follows. Using the notation of (4.3), attention to the proof of Proposition 12 shows that, for p in the range (4.11),

$$\int_{S(R)} |\varphi_r|^p\, dv \leqslant c_p e^{[N(1-(p/2))-(p/2)\sqrt{N^2-4E_r}]R}, \tag{4.31}$$

the exponent being negative for p in this range. Then, from the estimate (4.28), we obtain the following result.

PROPOSITION 17. *Under the hypotheses of Proposition* 14, *suppose furthermore that all points in $B_R(x)$ have images in M at distance $\geqslant T$ from x_0; i.e., the distance between the images of x_0 and x in M is $\geqslant R+T$. Then*

$$|\varphi_r(x)| \leqslant c_p[v(x, R)\, e^{N(T-R)}]^{1/p} e^{(1/2)[N-\sqrt{N^2-4E_r}](R-T)} \tag{4.32}$$

for p in the range (4.11).

It is reasonable to regard Propositions 14 and 17 as first steps toward an estimate whose final form is not yet clear. Particularly for quotients of $\mathbf{H}^{N+1}$ which possesses cusps of intermediate rank, deceding to what degree the sorts of results presented here extend presents interesting problems.

One final result we can obtain on the eigenfunctions φ_r for general geometrically finite domains is the following generalization of (3.2), which complements Proposition 12.

Proposition 18. *If φ_r is an L^2 eigenfunction with eigenvalue $E_r < N^2/4$, then*

$$\varphi_r \in L^p(M) \quad \text{for} \quad p \in [2, 2/[1 - \sqrt{1 - 4E_r/N^2}]). \tag{4.33}$$

For the proof, if M^* is any region in M on which the injectivity radius is $\geqslant C_0 > 0$, then $\varphi_r \in L^\infty(M^*)$, and hence $\varphi_r|_{M^*}$ belongs to $L^p(M^*)$ for each $p \in [2, \infty)$. It remains to examine φ_r restricted to the complement of such a region. In the case under consideration, it then suffices to consider the restriction of φ_r to a region near a cusp, say of rank $N-k$, which, after applying an appropriate automorphism to $\mathbf{H}^{N+1}$, is of the form

$$M^b = \Omega \times [1, \infty), \tag{4.34}$$

where Ω is an intersection of a number of half-spaces in $\mathbf{R}^N$. We can suppose M^b contains the point $(0, 1)$. If we denote coordinates on (4.34) by (x, y), then by estimating the number of fundamental domains a unit ball centered at a point in M^b intersects, as in the proof of Proposition 13, we obtain an estimate

$$|\varphi_r(x, y)|^2 \leqslant C y^{N-k} \left(\frac{y}{|x|^2 + y^2 + 1} \right)^{\sqrt{N^2 - 4E_r}}, \tag{4.35}$$

if M^b corresponds to a cusp of rank $N-k$.

Proof. If we apply the estimate

$$\int_X |\varphi_r|^p \, dv \leqslant \int_X |\varphi_r|^2 \, dv \, \|\varphi_r\|^{p-2}_{L^\infty(X)} \tag{4.36}$$

to regions of the form $e^j \leqslant y \leqslant e^{j+1}$ within M^b, and use (4.4) to estimate the square integral, we obtain an estimate

$$\int_{M^b} |\varphi_r|^p \, dv \leqslant c \int_1^\infty y^{-\sqrt{N^2 - 4R_r}} y^{((p-2)/2)(N-k-\sqrt{N^2-4E_r})} y^{-1} \, dy \tag{4.37}$$

which is finite as long as

$$\frac{p}{2}(N - k - \sqrt{N^2 - 4E_r}) < N - k, \tag{4.38}$$

or equivalently

$$\frac{p}{2}(N - \sqrt{N^2 - 4E_r}) < N + \left(\frac{p}{2} - 1 \right) k. \tag{4.39}$$

This establishes (4.33), and proves the proposition. ∎

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